Jordan algebra toolbox
From H-atom to Landau problem

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Outlines

- Jordan Algebras $\mathfrak{J}$

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- Majorana spinors and $Sp(4, \mathbb{R})$

Articles

- Tekin Dereli, Philippe Nounahon, Todor Popov
  A Remarkable Dynamical Symmetry of
  the Landau Problem.
  IoP Tekin Dereli's Festschrift, Istanbul 2021
- Mariana Kirchbach, Todor Popov, José-Antonio Vallejo.
  Color confinement at the boundary of AdS_5.
  Journal of High Energy Physics, 2021(9).

Euclidean Jordan algebra $\mathfrak{J}$

Algebra of observables in Quantum Mechanics
(Pascual Jordan)

$$x \circ y = y \circ x = \frac{1}{2}(xy + yx)$$

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x) \qquad\qquad \forall x, y \in \mathfrak{J} \, .$$

$\mathfrak{J}$ is commutative but is not associative
Euclidean Jordan algebra (Jordan, von Neumann, Wigner)

$$x^2 + y^2 = 0 \qquad \Rightarrow x = 0 \quad \text{and} \quad y = 0$$

it is power-associativity, $x^{m+n} = x^m \circ x^n$

Tits-Kantor-Koecher Construction

conformal (Möbius) algebra

$$\mathfrak{co}(\mathfrak{J}) = \mathfrak{g}_{+1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \cong \mathfrak{J} \oplus \mathfrak{str}(\mathfrak{J}) \oplus \mathfrak{J}^*$$

$\mathfrak{co}(\mathfrak{J})$ is a 3-graded Lie algebra with involution $\dagger$

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \qquad \mathfrak{g}_k^\dagger = \mathfrak{g}_{-k}$$

$$(x, y, z) := [[x, y^\dagger], z] \qquad x, y, z \in \mathfrak{g}_{+1} \cong \mathfrak{J} \ .$$

Jordan algebra and Jordan triple product

$$(abc) = a \circ (b \circ c) - b \circ (a \circ c) + (a \circ b) \circ c \,.$$

Jordan Triple System

$$\begin{aligned} (abc) &= (cba) \qquad (1) \\ (ab(cdx)) - (cd(abx)) &= (a(dcb)x) - ((cda)bx) \,. \end{aligned}$$

a linear map $S_x^y : \mathfrak{J} \to \mathfrak{J}$ through

$$S_x^y(z) = (xyz) \,.$$

Structure algebra $\mathfrak{str}(\mathfrak{J})$

$$[S_a^b, S_c^d] = S_{(abc)}^d - S_c^{(dab)} = S_a^{(bcd)} - S_{(cda)}^b \qquad (2)$$

TKK conformal algebra $\mathfrak{co}(\mathfrak{J})$

Structure constants $\Sigma_{\mu\rho}^{\nu\sigma}$

$$S_x^y(z) = (xyz) \qquad (e_\mu, e_\nu, e_\rho) = \Sigma_{\mu\rho}^{\nu\sigma} e_\sigma \ .$$

determine conformal algebra $\mathfrak{co}(\mathfrak{J}) = \mathfrak{g}_{+1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \cong \mathfrak{J}$

$$\begin{array}{lllllll} [U_a, U^b] & = & -2S_a^b \,, & [U_a, U_b] & = & 0 \,, & U_a \in \mathfrak{g}_{-1} \\ [S_a^b, U_c] & = & U_{(abc)} \,, & [S_a^b, S_c^d] & = & S_{(abc)}^d - S_c^{(bad)} \,, & S_a^b \in \mathfrak{g}_0 \\ [S_a^b, U^c] & = & -U^{(bac)} \,, & [U^a, U^b] & = & 0 \,, & U^b \in \mathfrak{g}_{+1} \end{array}$$

Guowu Meng $\mathfrak{J}$-Kepler problem

Jordan Triple Structure constants

$$x = x^{\mu}\sigma_{\mu} = x = x^{\mu}\sigma_{\mu} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \qquad x \in \mathfrak{J}_2^{\mathbb{C}}$$

$$(\sigma_\alpha, \sigma_\beta, \sigma_\gamma) = \Sigma^{\beta\rho}_{\alpha\gamma}\sigma_\rho = \sigma_\alpha\circ(\sigma_\beta\circ\sigma_\gamma) - \sigma_\beta\circ(\sigma_\gamma\circ\sigma_\alpha) + (\sigma_\alpha\circ\sigma_\beta)\circ\sigma_\gamma$$

gives back the concise formula

$$\Sigma^{\beta\rho}_{\alpha\gamma} = \delta^{\rho}_{\gamma}\delta^{\beta}_{\alpha} + \delta^{\rho}_{\alpha}\delta^{\beta}_{\gamma} - g^{\beta\rho}g_{\alpha\gamma} \; . \tag{3}$$

Günaydın's Oscillator Realization of $\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2,4)$

$\mathfrak{co}(\mathfrak{J})$	operator	$\in \mathfrak{co}(\mathfrak{J})$	mapping	$x-$rep basis $\mathfrak{so}(2,4)$
$\mathfrak{J}$	$U_a = -ia^\mu P_\mu$	$\in \mathfrak{g}_{-1}$	$x \mapsto a$	$P_\nu = i\partial_\nu$
$\mathfrak{str}(\mathfrak{J})$	$S_a^b = ia^\nu b_\mu S_\nu^\mu$	$\in \mathfrak{g}_0$	$x \mapsto (a,b,x)$	$S_\nu^\mu = -i\Sigma_{\nu\alpha}^{\mu\beta} x^\alpha \partial_\beta$
$\mathfrak{J}^*$	$U^b = ib_\mu K^\mu$	$\in \mathfrak{g}_{+1}$	$x \mapsto -(x,b,x)$	$K^\mu = i\Sigma_{\nu\alpha}^{\mu\beta} x^\nu x^\alpha \partial_\beta$

Jordan algebra $\mathfrak{J}_2^{\mathbb{C}}$ and 3-graded Lie algebra $\mathfrak{so}(2,4)$

$\mathfrak{g}_{-1}$	$-iP_\nu$	$=$	∂_ν	translations
$\mathfrak{g}_0$	$iM^\mu{}_\nu$	$=$	$-x^\mu \partial_\nu + x_\nu \partial^\mu$	Lorentz transformations
$\mathfrak{g}_0$	iD	$=$	$x^\mu \partial_\mu$	dilatation
$\mathfrak{g}_{+1}$	iK^μ	$=$	$-2x^\mu x^\nu \partial_\nu + x^\nu x_\nu \partial^\mu$	special conformal

Conformal algebra of Minkowski spacetime $\mathbb{R}^{1,3}$

$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2,4)$ acting on $\mathcal{M}_{1,3} = \mathcal{N}/\mathbb{R}^* \cong (S^1 \times S^3)/\mathbb{Z}_2$

$$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2,4) = \underbrace{(\mathfrak{J}_2^{\mathbb{C}})^*}_{K^\mu} \oplus \overbrace{(\underbrace{\mathfrak{so}(1,3)}_{M^\mu_\nu} \oplus \underbrace{\mathbb{R}}_{D})}^{\mathfrak{str}(\mathfrak{J}_2^{\mathbb{C}})} \oplus \underbrace{\mathfrak{J}_2^{\mathbb{C}}}_{P_\nu} \qquad \mu,\nu = 0,1,2,3$$

$$x = x^\mu \sigma_\mu = x = x^\mu \sigma_\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \qquad x \in \mathfrak{J}_2^{\mathbb{C}}$$

$$\Sigma^{\beta\rho}_{\alpha\gamma} = \delta^\rho_\gamma \delta^\beta_\alpha + \delta^\rho_\alpha \delta^\beta_\gamma - g^{\beta\rho} g_{\alpha\gamma} \,. \qquad (4)$$

many layer realization of the Erlangen program

$$Aut(\mathfrak{J}) \subset Str(\mathfrak{J}) \subset Co(\mathfrak{J})$$

$$SO(3) \subset SO(1,3) \subset SO(2,4)$$

Group	Transformations		Space
$Aut(\mathfrak{J})$	rotations	$x' = Ox$	eulidean
$Str_0(\mathfrak{J})$	Lorentz group	$x' = \Lambda x$	pseudo-euclidean
$\mathcal{P}(\mathfrak{J})$	Poincaré group	$x' = \Lambda x + a$	affine pseudo-euclid
$\mathcal{P}_+(\mathfrak{J})$	Similitude group	$x' = e^{\rho}(\Lambda x + a)$	rescaled affine pseu
$Co(\mathfrak{J})$	Möbius group	$z' = \frac{az+b}{cz+d}$	projective

acting on projective space of rays in the Hilbert space

$$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) = \mathfrak{so}(2,4) = \underbrace{(\mathfrak{J}_2^{\mathbb{C}})^*}_{K^\mu} \oplus \overbrace{(\underbrace{\mathfrak{so}(1,3)}_{M_\nu^\mu} \oplus \underbrace{\mathbb{R}}_{D})}^{\mathfrak{str}(\mathfrak{J}_2^{\mathbb{C}})} \oplus \underbrace{\mathfrak{J}_2^{\mathbb{C}}}_{P_\nu}$$

TKK for Jordan algebra $\mathfrak{J}_2^{\mathbb{R}} = H_2(\mathbb{R})$

real Pauli matrices with coordinates $\{y_0, y_1, y_2\}$

$$y = \sum_{\mu=0,1,3} x^\mu \sigma_\mu = \begin{pmatrix} y_0 + y_2 & y_1 \\ y_1 & y_0 - y_2 \end{pmatrix}, \quad y^T = y\,. \qquad (5)$$

conformal algebra of Minkowski space $\mathbb{R}^{1,2}$ acting on $\mathcal{M}_{1,2} = \mathcal{N}/\mathbb{R}^* \cong (S^1 \times S^2)/\mathbb{Z}_2$

$$\mathfrak{co}(\mathfrak{J}_2^{\mathbb{R}}) = \mathfrak{so}(2,3) = \underbrace{(\mathfrak{J}_2^{\mathbb{R}})^*}_{K^{\tilde{\mu}}} \oplus \overbrace{(\underbrace{\mathfrak{so}(1,2)}_{M^{\tilde{\mu}}_{\tilde{\nu}}} \oplus \underbrace{\mathbb{R}}_{D})}^{\mathfrak{str}(\mathfrak{J}_2^{\mathbb{R}})} \oplus \underbrace{\mathfrak{J}_2^{\mathbb{R}}}_{P_{\tilde{\nu}}} \qquad \tilde{\mu}, \tilde{\nu} = 0, 1, 2$$

Conformal Time Transfromations $SO(1,2)$

TKK construction of the Jordan algebra $\mathfrak{J}_1^{\mathbb{R}} = \mathbb{R}$

$$\mathfrak{co}(\mathbb{R}) = \mathfrak{so}(1,2) = (\mathfrak{J}_1^{\mathbb{R}})^* \oplus \mathfrak{str}(\mathfrak{J}_1^{\mathbb{R}}) \oplus \mathfrak{J}_1^{\mathbb{R}} = \mathbb{R}K_0 \oplus \mathbb{R}D \oplus \mathbb{R}P_0 \ .$$

different $SO(1,2)$-generator is a conformal hamiltonian:

$$\begin{array}{rcl} B_0 &=& r(p^2+1)/2 \qquad E<0 \quad \text{bound states dS ,} \\ A_0 &=& r(p^2-1)/2 \qquad E>0 \quad \text{scattering states AdS ,} \\ A_0+B_0 &=& rp^2 \qquad E=0 \quad \text{free motion Mink .} \end{array}$$

Newton-Hooke duality

Hydrogen atom versus Harmonic Oscillators

$$e^- \text{ in electric field} \quad \overset{dual}{\leftrightarrow} \quad e^- \text{ in magnetic field} \; .$$

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Kustaanheimo-Stiefel Conformal spinorial regularization
Barut & co, Mack & Todorov

3D H-atom	$\leftrightarrow$	4D Harmonic Oscillator
$SO(2,4)$	$\cong$	$SU(2,2)$

Newton-Hooke duality

Hydrogen atom versus Harmonic Oscillators

$$e^- \text{ in electric field} \overset{dual}{\leftrightarrow} e^- \text{ in magnetic field} \; .$$

Kustaanheimo-Stiefel Conformal spinorial regularization
Barut & co, Mack & Todorov

3D H-atom	$\leftrightarrow$	4D Harmonic Oscillator
$SO(2,4)$	$\cong$	$SU(2,2)$
2D H-atom	$\leftrightarrow$	2D Harmonic Oscillator
$SO(2,3)$	$\cong$	$Sp(4,\mathbb{R})$ (Landau problem)

Levi-Civita (Bohlin) regularization
Dirac "A Remarkable Representation of the 3 + 2 de Sitter Group"

H-atom Spectrum Generating Algebra $SO(2,4)$

Dynamical symmetry $SO(2,4)$ of H-atom spectrum
(Malkin & Manko 1966; Barut and co)

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{r} \times \boldsymbol{p} & B_0 - A_0 &= r & B_0 + A_0 &= r\boldsymbol{p}^2 \\ \boldsymbol{\Gamma} &= r\boldsymbol{p} & \boldsymbol{B} - \boldsymbol{A} &= \boldsymbol{r} & \boldsymbol{B} + \boldsymbol{A} &= \boldsymbol{r}\boldsymbol{p}^2 - 2\boldsymbol{p}(\boldsymbol{r} \cdot \boldsymbol{p}) \\ D &= \boldsymbol{r} \cdot \boldsymbol{p} - i \end{aligned}$$

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$$
\begin{array}{rclrclrcl}
\boldsymbol{L} & = & \boldsymbol{r} \times \boldsymbol{p} & B_0 - A_0 & = & r & B_0 + A_0 & = & r\boldsymbol{p}^2 \\
\boldsymbol{\Gamma} & = & r\boldsymbol{p} & \boldsymbol{B} - \boldsymbol{A} & = & \boldsymbol{r} & \boldsymbol{B} + \boldsymbol{A} & = & \boldsymbol{r}\boldsymbol{p}^2 - 2\boldsymbol{p}(\boldsymbol{r}\cdot\boldsymbol{p}) \\
D & = & \boldsymbol{r}\cdot\boldsymbol{p} - i & & & & & &
\end{array}
$$

Causal space-time automorphisms of the Minkowski space $\mathbb{R}^{1,3}$

$$
\begin{array}{rcll}
[K_\mu, P_\nu] & = & 2i(g_{\mu\nu}D - M_{\mu\nu})\ , & [D, P_\mu] = iP_\mu\ , \\
[K_\lambda, M_{\mu\nu}] & = & i(g_{\lambda\mu}K_\nu - g_{\lambda\nu}K_\mu)\ , & [D, M_{\mu\nu}] = 0\ , \\
[P_\lambda, M_{\mu\nu}] & = & i(g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu)\ , & [D, K_\mu] = -iK_\mu\ .
\end{array}
$$

$\mathfrak{co}(\mathfrak{J})$	space-time cone	hydrogen atom
$\mathfrak{g}_0$	$M_{\mu\nu}$; D	$L_{ij}, \Gamma_i; D$
$\mathfrak{g}_{-1}$	K_μ	$B_\mu + A_\mu$
$\mathfrak{g}_{+1}$	P_μ	$B_\mu - A_\mu$

.

A_i Laplace-Runge-Lenz vector, its dual partner B_i

H-atom SGA $SO(2,4)$

Conformal algebra of Minkowski $\mathbb{R}^{1,3}$

$$L_{ab} \in \mathfrak{so}(2,4)\ , \qquad \eta_{ab} = diag(+1,+1,-1,-1,-1,-1)\ .$$

$$[L_{ab}, L_{cd}] = -i(\eta_{ac}L_{bd} + \eta_{bd}L_{ac} - \eta_{ad}L_{bc} - \eta_{bc}L_{ad}) \qquad (6)$$

$$L_{ab} = \begin{pmatrix} 0 & B_0 & B_1 & B_2 & B_3 & D \\ & 0 & \Gamma_1 & \Gamma_2 & \Gamma_3 & A_0 \\ & & 0 & L_3 & -L_2 & A_1 \\ & & & 0 & L_1 & A_2 \\ & & & & 0 & A_3 \\ & & & & & 0 \end{pmatrix} \qquad a, b \in \{-1, 0, 1, 2, 3, 5\}\ .$$

accidental symmetry $SO(4)$ generated by $\boldsymbol{L}$ and $\boldsymbol{A}$
Vladimir A. Fock (1935)

Fock's sphere and null cone

Null cone of isotropic rays in $\mathbb{R}^{2,4}$

$$\mathcal{N} = \{\vec{x} \in \mathbb{R}^{2,4} | x_{-1}^2 + x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_5^2 = 0; x \neq 0\}.$$

compactified null cone $\mathcal{M}_{1,3} = \mathcal{N}/\mathbb{R}^* \cong (S^1 \times S^3)/\mathbb{Z}_2$

$$p_0 = -2\mu E > 0$$

Bander Itzykson 1968
stereographic projection to S^3 stable under $SO(4)$

$$\boldsymbol{u} = \frac{p^2 - p_0^2}{p^2 + p_0^2}\boldsymbol{n} + \frac{2p_0}{p^2 + p_0^2}\boldsymbol{p}$$

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hyperboloid stablized by $SO(1,3)$ generated by $\boldsymbol{L}$ and $\boldsymbol{B}$

2D H-atom SGA $SO(2,3)$

Conformal algebra of Minkowski $\mathbb{R}^{1,2}$

$$L_{ab} = \begin{pmatrix} 0 & B_0 & B_1 & B_2 & D \\ & 0 & \Gamma_1 & \Gamma_2 & A_0 \\ & & 0 & L_3 & A_1 \\ & & & 0 & A_2 \\ & & & & 0 \end{pmatrix} \qquad a, b \in \{-1, 0, 1, 2, 3 = 5\}$$

Null Cone of isotropic rays in $\mathbb{R}^{2,3}$

$$\tilde{\mathcal{N}} = \{\vec{y} \in \mathbb{R}^{2,3} | y_{-1}^2 + y_0^2 - y_1^2 - y_2^2 - y_3^2 = 0; y \neq 0\}$$

Compactified null cone

$$\mathcal{M}_{1,2} = \tilde{\mathcal{N}} / \mathbb{R}^* \cong (S^1 \times S^2)/\mathbb{Z}_2$$

Landau Problem and Harmonic oscillator

Larmour frequency $\omega = \frac{eB}{mc}$

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{\mathcal{A}}\right)^2 =: \frac{1}{2m}\boldsymbol{P}^2 \ . \qquad (7)$$

$$\boldsymbol{\mathcal{A}} = (\mathcal{A}_x, \mathcal{A}_y) = \frac{B}{2}(-y, x) \ , \qquad \mathcal{A}_i = -\frac{B}{2}\epsilon_{ij}x^j \qquad (8)$$

$$H = \frac{P_x^2 + P_y^2}{2m} = \frac{\hbar\omega}{4}\left\{p_\xi^2 + p_\eta^2 + (\xi^2 + \eta^2)\right\} - \frac{\hbar\omega}{2}(\xi p_\eta - \eta p_\xi) \ .$$

$$H = \frac{\hbar\omega}{2}\{a^+, a^-\} \ , \qquad [H, a^\pm] = \pm a^\pm \ , \qquad [H, b^\pm] = 0 \ . \qquad (9)$$

Dirac dynamical algebra $\mathfrak{so}(2,3) \cong \mathfrak{sp}(4,R)$

Dirac's realization of $\mathfrak{so}(2,3)$
"A Remarkable Representation of the 3 + 2 de Sitter Group"

$$
\begin{aligned}
m_{12} &= \tfrac{1}{2}(z\partial - \bar{z}\bar{\partial}) &&= \tfrac{1}{4}\left(\{b^-,b^+\} - \{a^-,a^+\}\right),\\
m_{23} &= \tfrac{1}{16}(z^2+\bar{z}^2) - (\partial^2+\bar{\partial}^2) &&= \tfrac{1}{4}\left(\{a^-,b^+\} + \{a^+,b^-\}\right),\\
m_{31} &= \tfrac{i}{16}(z^2-\bar{z}^2) + i(\partial^2-\bar{\partial}^2) &&= \tfrac{i}{4}\left(\{a^-,b^+\} - \{a^+,b^-\}\right),\\
m_{1-1} &= \tfrac{-i}{16}(z^2-\bar{z}^2) + i(\partial^2-\bar{\partial}^2) &&= \tfrac{i}{4}\left(a^+a^+ - a^-a^- + b^-b^- - b^+b^+\right),\\
m_{2-1} &= \tfrac{1}{16}(z^2+\bar{z}^2) + (\partial^2+\bar{\partial}^2) &&= \tfrac{1}{4}\left(a^-a^- + a^+a^+ + b^-b^- + b^+b^+\right),\\
m_{3-1} &= -\tfrac{i}{2}\,(z\partial + \bar{z}\bar{\partial} - 1) &&= -\tfrac{i}{4}\left(\{a^-,b^-\} - \{a^+,b^+\}\right),\\
m_{01} &= -\tfrac{1}{2}\,(z\bar{\partial} - \bar{z}\partial) &&= -\tfrac{1}{4}\left(a^-a^- + a^+a^+ - b^-b^- - b^+b^+\right),\\
m_{02} &= -\tfrac{i}{2}\,(\bar{z}\partial + z\bar{\partial}) &&= -\tfrac{i}{4}\left(a^-a^- - a^+a^+ + b^-b^- - b^+b^+\right),\\
m_{03} &= \tfrac{1}{8}z\bar{z} + (\partial\bar{\partial} + \bar{\partial}\partial) &&= -\tfrac{1}{4}\left(\{a^+,b^+\} + \{a^-,b^-\}\right),\\
m_{-10} &= \tfrac{1}{8}z\bar{z} - (\partial\bar{\partial} + \bar{\partial}\partial) &&= \tfrac{1}{4}\left(\{a^-,a^+\} + \{b^-,b^+\}\right).
\end{aligned}
$$

Landau Hamiltonian $\frac{H}{\hbar\omega} = m_{-10} - m_{12} = \frac{1}{2}\{a^-,a^+\}$

Angular momentum $\frac{L_z}{\hbar} = m_{12} \qquad \omega = \frac{eB}{mc}$

2D H-atom and Landau problem

$$
\begin{array}{ccccc}
\mathfrak{co}(\mathfrak{J}_2^{\mathbb{C}}) & = & \mathfrak{so}(2,4) & \xleftarrow{Kustaanheimo-Stiefel} & \mathfrak{su}(2,2) \\
\uparrow {\scriptstyle x=x^t} & & & & \uparrow {\scriptstyle \psi=\psi^c} \\
\mathfrak{co}(\mathfrak{J}_2^{\mathbb{R}}) & = & \mathfrak{so}(2,3) & \xleftarrow{Levi-Civita} & \mathfrak{sp}(4,\mathbb{R})
\end{array}
.
$$

2D e^- in electric field $\overset{Newton-Hooke}{\leftrightarrow}$ 2D e^- in magnetic field matrices in $\mathfrak{sl}(4,\mathbb{C})$ preserving a pseudo-Hermitian form β with signature $(++--)$

$$(\phi,\psi) = \phi_1^*\psi_1 + \phi_2^*\psi_2 - \phi_3^*\psi_3 - \phi_4^*\psi_4 = \phi^\dagger\beta\psi = \bar{\phi}\psi \ .$$

The Hermitian matrix $\beta = \beta^\dagger$ depends on the basis, it fixes the choice of the Dirac matrix $\gamma_0 := \beta$ and the invariance of the form implies

$$\beta\sigma^{AB}\beta^{-1} = (\sigma^{AB})^\dagger \ .$$

the 4×4 Dirac gamma matrices γ^μ,

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu} \qquad \mu,\nu = 0,1,2,3 \ .$$

The 15 $\mathfrak{su}(2,2)$-generators σ^{AB} $A,B \in \{-1,0,1,2,3,5\}$

$$\begin{array}{rclrcl} \sigma^{\mu\nu} &=& \frac{i}{4}[\gamma^\mu,\gamma^\nu]\ , & \sigma^{-15} &=& -\frac{1}{2}\gamma^5\ , \\ \sigma^{\mu5} &=& \frac{i}{4}[\gamma^\mu,\gamma^5]\ , & \sigma^{-1\mu} &=& -\frac{1}{2}\gamma^\mu\ . \end{array} \tag{10}$$

Here we denoted $\gamma^5 := \gamma^0\gamma^1\gamma^2\gamma^3$.

minimal (zero-mass) $U(2,2)$ representation

2 complex variables z^α and derivatives $w_\alpha = -i\frac{\partial}{\partial z^\alpha}$

$$\boldsymbol{z} = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \qquad \boldsymbol{\partial} = \begin{pmatrix} \frac{\partial}{\partial z^1} \\ \frac{\partial}{\partial z^2} \end{pmatrix} \qquad \psi = \begin{pmatrix} \bar{z} \\ \partial \end{pmatrix} \qquad \bar{\psi} = (-\bar{\partial}, z)\,. \tag{11}$$

A canonical representation of a pair of 2D harmonic oscillators

$$[\psi^\alpha, \bar{\psi}_\beta] = \delta^\alpha_\beta\,, \qquad [\psi^\alpha, \psi^\beta] = 0\,.$$

Mack & Todorov zero-mass $U(2,2)$ representation

$$J^{AB} = \bar{\psi}\sigma^{AB}\psi\,, \qquad C_1 = \bar{\psi}\psi$$

its restriction to Poincaré is irreducible
the helicities λ are labelling the zero-mass representations of $U(2,2)$.

$$C_1 + 2 = -2\lambda = z^\alpha \frac{\partial}{\partial z^\alpha} - \bar{z}^\alpha \frac{\partial}{\partial \bar{z}^\alpha} \qquad \lambda = 0, \pm\frac{1}{2}, \pm 1, \dots$$

The hydrogen atom is described by helicity $\lambda = 0$

Majorana reduction from $SU(2,2)$ to $Sp(4,\mathbb{R})$-rep (Stoyanov & Todorov)

$$\psi^c = \psi$$

$$\chi^\alpha = \begin{pmatrix} b^- \\ a^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z} + \partial \\ z + \bar{\partial} \end{pmatrix} , \quad \chi^*_\alpha = (b^+ \;\; a^+) .$$

2D H-atom $\mathfrak{so}(2,3)$-spinorial representation, Barut & Duru

$$\begin{aligned} m_{ij} &= \tfrac{1}{2}\epsilon_{ijk}\chi^*\sigma_k^T\chi , & m_{-1i} &= \tfrac{i}{4}(\chi^*\sigma_i^T\epsilon^T(\chi^*)^T - \chi^T\epsilon\sigma_i^T\chi) , \\ m_{-10} &= \tfrac{1}{2}(\chi^*\chi + 1) , & m_{0i} &= \tfrac{1}{4}(\chi^*\sigma_i^T\epsilon^T(\chi^*)^T + \chi^T\epsilon\sigma_i^T\chi) \end{aligned}$$

Dirac dynamical algebra $\mathfrak{so}(2,3) \cong \mathfrak{sp}(4,R)$

$Sp(4,\mathbb{R}) = Spin(2,3)$ acting on $S^1 \times S^2/\mathbb{Z}_2$

$$
\begin{aligned}
m_{12} &= \tfrac{1}{2}(z\partial - \bar{z}\bar{\partial}) &&= \tfrac{1}{4}\left(\{b^-, b^+\} - \{a^-, a^+\}\right) ,\\
m_{23} &= \tfrac{1}{16}(z^2 + \bar{z}^2) - (\partial^2 + \bar{\partial}^2) &&= \tfrac{1}{4}\left(\{a^-, b^+\} + \{a^+, b^-\}\right) ,\\
m_{31} &= \tfrac{i}{16}(z^2 - \bar{z}^2) + i(\partial^2 - \bar{\partial}^2) &&= \tfrac{i}{4}\left(\{a^-, b^+\} - \{a^+, b^-\}\right) ,\\
m_{1-1} &= \tfrac{-i}{16}(z^2 - \bar{z}^2) + i(\partial^2 - \bar{\partial}^2) &&= \tfrac{i}{4}\left(a^+a^+ - a^-a^- + b^-b^- - b^+b^+\right) ,\\
m_{2-1} &= \tfrac{1}{16}(z^2 + \bar{z}^2) + (\partial^2 + \bar{\partial}^2) &&= \tfrac{1}{4}\left(a^-a^- + a^+a^+ + b^-b^- + b^+b^+\right) ,\\
m_{3-1} &= -\tfrac{i}{2}\left(z\partial + \bar{z}\bar{\partial} - 1\right) &&= -\tfrac{i}{4}\left(\{a^-, b^-\} - \{a^+, b^+\}\right) ,\\
m_{01} &= -\tfrac{1}{2}\left(z\bar{\partial} - \bar{z}\partial\right) &&= -\tfrac{1}{4}\left(a^-a^- + a^+a^+ - b^-b^- - b^+b^+\right) ,\\
m_{02} &= -\tfrac{i}{2}\left(\bar{z}\partial + z\bar{\partial}\right) &&= -\tfrac{i}{4}\left(a^-a^- - a^+a^+ + b^-b^- - b^+b^+\right) ,\\
m_{03} &= \tfrac{1}{8}z\bar{z} + (\partial\bar{\partial} + \bar{\partial}\partial) &&= -\tfrac{1}{4}\left(\{a^+, b^+\} + \{a^-, b^-\}\right) ,\\
m_{-10} &= \tfrac{1}{8}z\bar{z} - (\partial\bar{\partial} + \bar{\partial}\partial) &&= \tfrac{1}{4}\left(\{a^-, a^+\} + \{b^-, b^+\}\right) .
\end{aligned}
$$

S^1 generated by conformal Hamiltonian $m_{-10} \in \mathfrak{so}(2)$
S^2 stabilized by $m_{12}, m_{23}, m_{31} \in \mathfrak{so}(3)$:
$2D$ Runge-Lenz vector (m_{23}, m_{31}) and angular momentum m_{12}

Reduction of Kustaanheimo-Steifel *KS* transform

In celestial mechanics *KS* removes the singular trajectories due to binary collisions from the phase space of 3D Kepler motion Kustaanheimo-Stiefel Conformal spinorial regularization

3D H-atom	$\leftrightarrow$	4D Harmonic Oscillator
$SO(2,4)$	$\cong$	$SU(2,2)$

elliptic Kepler orbits $E < 0$: geodesic motion on sphere S^3
Hodograph: stereographic projection of the momentum space (Fock 1935)

Hopf fibration and KS regularization

$$0 \to S^1 \hookrightarrow S^3 \to S^2 \to 0 \; ,$$

$$|\boldsymbol{x}| = |\boldsymbol{z}|^2 \; , \qquad \boldsymbol{z} = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} \; , \qquad |\boldsymbol{z}|^2 = \boldsymbol{z}^\dagger \boldsymbol{z} .$$

KS is cotangent extention of Hopf mapping (Bruno Cordani)

$$KS : \quad T^+ S^3 \to T^+ S^2 \qquad T^+ S^n = T^* S^n - \{0\}$$

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$$KS: \quad T^+S^3 \to T^+S^2 \qquad T^+S^n = T^*S^n - \{0\}$$

$$\begin{aligned} x_1 &= \quad u_1 u_3 + u_2 u_4 \,, \\ x_2 &= \quad u_2 u_3 - u_1 u_4 \,, \\ x_3 &= -u_1^2 - u_2^2 + u_3^2 + u_4^2 \,, \end{aligned}$$

Hopf fibration and KS regularization

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$$\begin{aligned} p_1 &= -(u_1 w_3 + w_1 u_3 + u_2 w_4 + w_2 u_4)/|\boldsymbol{z}|^2 \,, \\ p_2 &= -(u_2 w_3 + w_2 u_3 - w_1 u_4 - u_1 w_4)/|\boldsymbol{z}|^2 \,, \quad . \\ p_3 &= (u_1 w_1 + u_2 w_2 - u_3 w_3 - u_4 w_4)/|\boldsymbol{z}|^2 \,. \end{aligned}$$

The coordinates on T^+S^3 are subject to the constraint

$$K = u_1 w_2 - u_2 w_1 + u_3 w_4 - u_4 w_3 = 0 \,.$$

Majorana Spinor Reduction

$$|\boldsymbol{x}| = |\boldsymbol{z}|^2 , \qquad \boldsymbol{z} = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix} , \qquad |\boldsymbol{z}|^2 = \boldsymbol{z}^\dagger \boldsymbol{z}.$$

Planar reduction

$$u_2 = u_4 = 0 , \qquad w_2 = w_4 = 0$$

Levi-Civita regularization $T^+S^1 \to T^+S^1$

$$\xi + i\eta = Z^2 , \qquad \begin{array}{rcl} \xi & = & u_1^2 - u_3^2 , \\ \eta & = & 2u_1 u_3 , \end{array} \qquad \begin{array}{l} p_\xi = (u_1 w_1 - w_3 u_3)/|Z|^2 , \\ p_\eta = -(u_1 w_3 + w_1 u_3)/|Z|^2 \end{array}$$

real spinor $\psi = \begin{pmatrix} u_1 \\ u_3 \end{pmatrix}$ is written as $Z = u_1 + iu_3 \in \mathbb{C}$

$$0 \to S^0 \hookrightarrow S^1 \to S^1 \to 0 ,$$

$$\chi^\alpha = \begin{pmatrix} b^- \\ a^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z} + \partial \\ z + \bar{\partial} \end{pmatrix} , \quad \chi^*_\alpha = (b^+ \;\; a^+) \, .$$

2D H-atom $\mathfrak{so}(2,3)$-spinorial representation, Barut & Duru

$$\begin{aligned} m_{ij} &= \tfrac{1}{2}\epsilon_{ijk}\chi^*\sigma_k^T\chi \, , & m_{-1i} &= \tfrac{i}{4}(\chi^*\sigma_i^T\epsilon^T(\chi^*)^T - \chi^T\epsilon\sigma_i^T\chi) \, , \\ m_{-10} &= \tfrac{1}{2}(\chi^*\chi + 1) \, , & m_{0i} &= \tfrac{1}{4}(\chi^*\sigma_i^T\epsilon^T(\chi^*)^T + \chi^T\epsilon\sigma_i^T\chi) \end{aligned}$$

4D Dirac spinor yields 3D H-atom $\mathfrak{so}(2,4)$-spinorial representation

$$[\psi^\alpha, \bar{\psi}_\beta] = \delta^\alpha_\beta \, , \qquad [\psi^\alpha, \psi^\beta] = 0 \, .$$

via Mack & Todorov ladder $U(2,2)$ representation

$$J^{AB} = \bar{\psi}\sigma^{AB}\psi \, , \qquad C_1 = \bar{\psi}\psi$$

its Majorana reduction is $Sp(4,\mathbb{R})$-rep: Stoyanov & Todorov

$$\psi^c = \psi$$

Charge-Dyon system versus Haldane Spherical geometry for the Landau problem

3D MIC-Kepler	$\leftrightarrow$	4D Harmonic Oscillator
$SO(2,4)$	$\cong$	$SU(2,2), \lambda > 0$

Charge-Dyon system versus Haldane Spherical geometry for the Landau problem

3D MIC-Kepler	$\leftrightarrow$	4D Harmonic Oscillator
$SO(2,4)$	$\cong$	$SU(2,2), \lambda > 0$
2D MIC-Kepler	$\leftrightarrow$	2D Harmonic Oscillator
$SO(2,3)$	$\cong$	$Sp(4,\mathbb{R})$ (Haldane geometry)

Ter-Antonyan, Mardoyan, Nersessian
Dyon–Oscillator Duality
4D singular oscillator and generalized MIC-Kepler system
F. D. M. Haldane
Fractional Quantization of the Hall Effect: A Hierarchy of Incompressible Quantum Fluid States

Confinement of quarks and meson mass spectrum

Kirchbach, M., Popov, T., & Vallejo, J.A. (2021). Color confinement at the boundary of the conformally compactified AdS_5. Journal of High Energy Physics, 2021(9).

Alex Ganchev (1955-2022)

Connection to parabosons

n parabosons close $\mathfrak{osp}(1|2n)$, Ganchev, Palev

$$[\{b_i^+, b_j^-\}, b_k^+] = 2\delta_{jk} b_i^+ \qquad [\{b_i^+, b_j^+\}, b_k^+] = 0$$

$$\chi^\alpha = \begin{pmatrix} b^- \\ a^- \end{pmatrix} = \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix} \qquad \chi_\alpha^* = (b_1^+ \;\; b_2^+)$$

$n = 2$ parabosons $\mathfrak{osp}(1|4)$ Landau problem

$$\mathfrak{osp}(1|4)_0 = \mathfrak{sp}(4, \mathbb{R})$$

$n = 4$ parabosons $\mathfrak{sp}(8, \mathbb{R}) \subset \mathfrak{osp}(1|8)$ H-atom with spin

$$\mathfrak{osp}(1|8)_0 = \mathfrak{sp}(8, \mathbb{R})$$