Geometry of CR Submanifolds of Riemannian Manifolds

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Abstract

When a manifold is endowed with a geometric structure, we have more opportunities to explore its geometric properties. Following the Erlangen program of Klein, which claims that geometry is the investigation of properties which remain invariant under the action of a group, it is natural to investigate submanifolds of almost complex Hermitian manifolds $(\overline{M}, \overline{g}, J)$ which have a special behavior with respect to the almost complex structure J. Firstly, the almost complex submanifolds (where their tangent space is J-invariant) and totally real submanifolds (where their tangent space is J-anti-invariant) have been investigated.

A natural generalization of the above classes of submanifolds are the so-called CR submanifolds. A submanifold M of an almost complex Hermitian manifold $(\overline{M}, \overline{g}, J)$ is named a CR submanifold if there exist distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that $\mathcal{D} \oplus \mathcal{D}^{\perp} = TM$, $J\mathcal{D} = \mathcal{D}$, $J\mathcal{D}^{\perp} \subset T^{\perp}M$. Note that totally real submanifolds ($\mathcal{D} = \{0\}$) and almost complex submanifolds ($\mathcal{D} = TM$) are trivial examples of CR submanifolds. Moreover, real hypersurfaces of almost Hermitian manifolds are typical examples of CR submanifolds of maximal CR dimension.

Furthermore, the odd-dimensional analogue of CR submanifolds in Kählerian manifolds is the concept of contact CR submanifolds in Sasakian manifolds. Namely, a submanifold M in the Sasakian manifold $(\widetilde{M}, \varphi, \xi, \eta, \tilde{g})$ carrying a φ-invariant distribution $\mathcal{D}$, such that the orthogonal complement of $\mathcal{D}$ in TM is φ-anti-invariant, is called a contact CR submanifold.

One of the aims of submanifold geometry is to classify submanifolds according to given geometric data. For example, we study the relations and the interplay between intrinsic invariants, which only depend on the submanifold as a manifold itself, and extrinsic invariants, which depend on the immersion. As in Riemannian geometry the structure of a submanifold is encoded in the second fundamental form, we study CR submanifolds exploring the totally geodesic submanifolds and submanifolds close to them, characterizing several important classes of submanifolds with certain conditions on the submanifold structure (represented by the second fundamental form) and structure naturally induced from the ambient space.